The Dynamic Standing Wave Structure of the Hydrogen Atom

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Novel solutions for the s state statistical probability densities of the hydrogen atom are reported. The new probability densities result from a crucial feature of the hydrogen eigenfunctions, namely that they are the product of real functions from the time-independent Schrödinger equation and the complex wave e^(i ω_n t) from the full Schrödinger equation. The new statistical probability densities derive from standing wave patterns in the complex plane not fully appreciated as such generated by the full s state hydrogen eigenfunctions. The new statistical probability densities determined by the application of Born's rule have clear cyclic time dependence exhibiting spherically symmetric standing wave patterns in conventional coordinate physical space in the physical volume occupied by the hydrogen atom. This gives the hydrogen atom a dynamic structure not found in the literature where atomic structure in physical space is static with no standing waves.

I. Introduction

Max Born created a controversy in 1925[1] when he published a paper introducing what has become known as Born's rule: the probability density of a quantum entity, such as an electron bound to a proton by their mutual electric potential energy, is the product of the full complex eigenfunction of the bound electron multiplied by its complex conjugate. If in Schrödinger's equation the imaginary unit i is replaced by – i, complex conjugates are solutions. Schrödinger took vigorous exception to Born's rule, in its application to the eigenfunctions. Einstein rejected it as well opposing probabilistic interpretations.

Despite these negative reactions the physics community adopted Born's rule. Apparently Schrödinger imagined that the energy eigenfunctions' job was to produce energy eigenvalues, E_n (n is reserved for E, related eigenfunctions, and probability densities) and to attach no other physical import to the full eigenfunctions. Eigenvalues and eigenfunctions with physical content would be analogous to the solutions of classical resonant systems such as a taut cord. Fixed at both ends vibrating normal to the cord, the solutions of the equation of motion describe physical standing waves on the cord as well as their frequencies. Born's rule mathematized the indeterminacy that is characteristic of quantum physics introducing statistical probabilistic descriptions of quantum entities.

The complete hydrogen eigenfunction is a function of complex variables that includes a single, time dependent complex function $\Psi_{Tn}(t)$ that is a factor in the complete solution.[2,3] The full hydrogen eigenfunction can be placed on the complex plane where its time dependence and its statistical complex variable domain values can be followed for an electron in a bound state. Complete eigenfunctions produce rotating statistical standing wave patterns on the complex plane. Born's rule applied to the s state eigenfunctions results in the s state time dependent statistical probability densities of the hydrogen atom.

II. The s State Eigenfunctions of the Electron in the Hydrogen Atom

The s states of the hydrogen atom are convenient because they have minimum content but are general with a particular advantage: the s state energy eigenvalues E_n are a convergent infinite sequence of binding energies of the electron in the hydrogen atom (Bohr's energies), a significant pattern first identified indirectly in 1885 by Balmer[4]. Hydrogen atoms in p, d and f states have comparable resilience, size, stability and electron binding energies as determined from atomic spectra. However the s states show a path to limited generality without the distraction of additional details.

The long lifetime of electron states in the hydrogen atom allows the formation of the time-independent Schrödinger equation enabling a separation of the energy eigenfunctions in conventional coordinate physical space from a second function $\Psi_{Tn}(t)$ with complex variables. $\Psi_{Tn}(t)$ are the solution of the full Schrödinger equation. Since the n*th* stationary s state of hydrogen is sufficiently stable with a lifetime long compared to its cyclic period T_n , the Hamiltonian multiplying the energy eigenfunction $\Psi_{Rn}(r)$ can be set equal to the product of energy eigenvalue E_n and $\Psi_{Rn}(r)$ resulting in a classic eigenvalue differential equation system.[5] The energy eigenfunctions $\Psi_{Rn}(r)$ from the solution of this system are the associated Laguerre functions.[6] $\Psi_{Rn}(r)$ are functions of real variables, and are spherically symmetric. The spherical symmetry of the electric potential energy of the hydrogen atom is passed on to $\Psi_{Rn}(r)$ by Schrödinger's solutions to his time independent equation. As a consequence a thin shell of radius r and differential thickness dr centered very near the proton will have the same magnitude of $\Psi_{Rn}(r)$ throughout the thin shell independent of direction due to the spherical symmetry of the associated Laguerre functions. After E_n and $\Psi_{Rn}(r)$ are determined, the complex variable function of time $\Psi_{Tn}(t)$ can then be obtained by solving the full Schrödinger equation where the energy Hamiltonian is replaced by the product $E_n\ \Psi_{Rn}(r)$. Integrating the full Schrödinger equation then results in $\Psi_{Tn}(t) = e^{(i\, \omega_n t)}$ where $E_n = -(h/2\pi)\, \omega_n$. The complete eigenfunction for the hydrogen atom, $\Psi_n(r, t)$, is then the product of the real, spherically symmetric energy eigenfunctions $\Psi_{Rn}(r)$ and the time dependent complex function $\Psi_{Tn}(t)$:

$$\Psi_n(r, t) = \Psi_{Rn}(r)\, \Psi_{Tn}(t) = \Psi_{Rn}(r)\, e\wedge(i\, \omega_n\, t) = \Psi_{Rn}(r)\, [\cos(\omega_n\, t) + i \sin(\omega_n\, t)] \qquad (1)$$

where $\omega_n > 0$.[8] A feature of this solution is that function factor $\Psi_{Tn}(t) = e\wedge(i\, \omega_n\, t)$ governs the time evolution of $\Psi_n(r, t)$ producing standing wave patterns on the complex plane, a requirement for repetitive bound states. Eq.(1) is the full eigenfunction of the s states of the hydrogen atom as found in the literature.[8]

III. $\Psi_n(r, t)$ On The Complex Plane

Eq.(1) shows that $\Psi_n(r, t)$ is the product of $\Psi_{Rn}(r)$ and $\Psi_{Tn}(t)$. In classical physics the function $\Psi_{Tn}(t)$ is known as a unit phasor.[9] Eq.(1) shows that phasors play a role in quantum physics also. Additionally since $\Psi_{Rn}(r)$ are real functions, $\Psi_n(r, t)$ is also a phasor: $\Psi_n(r, t)$ is a real function $\Psi_{Rn}(r)$ multiple of the unit phasor $\Psi_{Tn}(t)$ in the complex plane. The complex unit phasor $\Psi_{Tn}(t)$ travels in anticlockwise motion as a point on the unit circle centered on the origin of the complex plane. The unit phasor $\Psi_{Tn}(t)$ can be conceptualized alternately as a unit complex vector with tail at the origin and head at the point $e\wedge(i\, \omega_n\, t)$ on the unit circle at time t. As time t evolves vector $\Psi_{Tn}(t)$ rotates about the origin of the complex plane, its tail fixed at the origin and its tip tracing out the unit circle on the complex plane with angular frequency ω_n. From Eq(1) $\Psi_{Tn}(t)$ drives all points of $\Psi_n(r, t)$ in circular paths in the complex plane in exact synchrony with $\Psi_{Tn}(t)$.

$\Psi_{Rn}(r)$ sets the physical scale of the phasor $\Psi_n(r, t)$. Since the late 1920's algebraic expressions of energy eigenfunctions[10] $\Psi_{Rn}(r)$ and graphical renditions of $\Psi_{Rn}(r)$ were produced in the literature. The graphs of the hydrogen energy eigenfunctions $\Psi_{Rn}(r)$ are found as $\Psi_n(r, 0)$ in the notation of Eq.(1) above. A survey of the literature for analytic expressions for the first ten $\Psi_{Rn}(r)$ was found in Ref. 3, page 243. There are multiple sources of the graphs of $\Psi_{Rn}(r)$ as a function of r including Ref. 2, page 530, Ref. 12, Fig. 21-3, page 142 and Ref. 13, Fig.4, page 16. $\Psi_{Rn}(0)$ is very close to the proton. Graphs of $\Psi_{Rn}(r)$ include the first three s states of hydrogen with n values 1, 2, and 3. $\Psi_{R1}(r)$ does not cross the r-axis but $\Psi_{R2}(r)$ crosses it once and $\Psi_{R3}(r)$ crosses it twice. $\Psi_{Rn}(r)$ falls rapidly towards 0 for values of r greater than the physical extent of the hydrogen atom for all states including n values of 1, 2, and 3. The length of phasor $\Psi_n(r, t)$ in the complex plane is the value of r where $\Psi_{Rn}(r)$ in the limit vanishes beyond the effective radius of the hydrogen atom for any given s state. $\Psi_{Rn}(r)$ determines where the tip of vector phasor $\Psi_n(r, t)$ occurs in the complex plane.

The values of $\Psi_{Rn}(r)$ at position r on the phasor $\Psi_n(r, t)$ from r = 0 to the tip of $\Psi_n(r, t)$ are the statistical density of the complex numbers that are the domain of the full eigenfunction $\Psi_n(r, t)$ at time t. These statistical density patterns are subject to Born's rule. Annular bands of domain with amplitudes $\Psi_{Rn}(r)$ are formed between the zeros of $\Psi_{Rn}(r)$ as in the figures referenced above where alternate annular bands of $\Psi_{Rn}(r)$ have opposite sign. The absolute value of the amplitude maximum for an annular band is roughly mid-way across that annular band. The motion of phasor $\Psi_n(r, t)$ in its anticlockwise rotation about the origin of the complex plane results in the various real values $\Psi_{Rn}(r)$ at position r in real physical space multiplying the complex plane coordinates to form an instantaneous statistical complex domain density values in the complex plane at all values of r in $\Psi_n(r, t)$ at time t. The standing wave pattern of complex domain density values sweeping around the complex plane resulting from the application of the mathematical recipe for that complex domain density from $\Psi_{Rn}(r)$ is determined and driven by $\Psi_n(r, t)$ in the complex plane.

The development of $\Psi_{Tn}(t)$ above as part of the solution of the full Schrödinger equation resulting in Eq.(1) commonly skips consideration of a constant of integration. The general solution for $\Psi_{Tn}(t)$ of the full Schrödinger equation is $\Psi_{Tn}(t) = e^{(i \omega_n t + C)}$ where C is the constant of integration. C sets the angular starting position $C = \omega_n t_C$ where time t_C sets the angle of the complex unit phasor $\Psi_{Tn}(t)$ in the complex plane at t = 0. C is an example of the kind of information that is often unavailable for quantum systems: it can be an indeterminate, unknowable magnitude. It is customary to set C and t_C to the value 0 without mention as illustrated above. C = 0 in Eq.(1). For a classical phasor, C is set by a knowable boundary condition.[9]

IV. Finding $\Psi_{Tn}(t)\ \Psi_{Tn}(t)^*$ and the Electron Probability Density $\Psi_n(r, t)\ \Psi_n(r, t)^*$

Eq.(1) are solutions of the Schrödinger equation where $\Psi_n(r, t)$ is the product of energy eigenfunction $\Psi_{Rn}(r)$, a real function, and the unit phasor $\Psi_{Tn}(t)$, a complex function. Following Born's rule since the mid-1920's this result and its complex conjugate has been used to find the electron probability density, the product $\Psi_n(r, t)\ \Psi_n(r, t)^*$. For anticlockwise $\Psi_{Tn}(t) = e^{(i \omega_n t)}$ and for clockwise complex conjugate $\Psi_{Tn}(t)^* = e^{(-i \omega_n t)}$, ordinary multiplication of $\Psi_{Tn}(t)$ and $\Psi_{Tn}(t)^*$ where the exponents add to 0 results in $\Psi_{Tn}(t)\ \Psi_{Tn}(t) = 1$.[8] But $\Psi_{Tn}(t) = \cos(\omega_n t) + i \sin(\omega_n t)$ and $\Psi_{Tn}(t)^* = \cos(\omega_n t) - i \sin(\omega_n t)$. Complex number multiplication yields $\Psi_n(r, t)\ \Psi_n(r, t)^* = [\cos(\omega_n t)]^2 + [\sin(\omega_n t)]^2 = 1$. The trigonometric expressions for $\Psi_{Tn}(t)$ and $\Psi_{Tn}(t)^*$ make it clear that there are infinitely many angles for an infinity of t values on one sweep around the unit circle where the two unit phasors are rotating position vectors.

With the result directly above the electron is in a state of suspended animation since there is no time dependence for $\Psi_{Tn}(t)\ \Psi_{Tn}(t)^*$ and thus no time dependence for the product $\Psi_n(r, t)\ \Psi_n(r, t)^*$ as well. Eq.(1) is contradicted: the electron's motion is determined by the time dependence of $\Psi_{Tn}(t)$ which produces standing wave patterns in the complex plane. The statistical probability density of the electron must be time dependent. The electron in a hydrogen atom is moving: a differential operator in the time-independent Hamiltonian for the hydrogen atom represents the kinetic energy of the electron. There is no knowable trajectory for the electron, but it moves with changing kinetic and electric potential energy in the spherically symmetric electric field of the proton.

The determination of the electron probability of the bound electron in the hydrogen atom is the energy eigenfunction $\Psi_{Rn}(r)$ from Schrödinger's time independent equation is a real function, not a complex function. This fact determines how the application of Born's rule is executed to form the product $\Psi_n(r, t)\ \Psi_n(r, t)^*$. Born's rule requires $\Psi_n(r, t)\ \Psi_n(r, t)^*$ to be real and positive. Since $\Psi_{Rn}(r)$ are real, Born's rule then requires that the product $\Psi_{Tn}(t)\ \Psi_{Tn}(t)^*$ must be real and positive confining that product to the real axis on the complex plane. Thus $\Psi_{Tn}(t)\ \Psi_{Tn}(t)^*$ is restricted to values on the real axis of the complex plane anywhere between and including the complex plane coordinates (1, 0 i) and (– 1, 0 i). In the initial trial considered here C is assigned the usual default value 0 for each unit phasor. Unit phasors $\Psi_{Tn}(t)$ and $\Psi_{Tn}(t)^*$ begin rotation with tips at the same point (1, 0 i) at t = 0. The rotation of the two unit phasors is coordinated by their shared time evolution due to their having the same magnitude of angular speed ω_n. At any given time t, the real coordinates of the unit phasors on the real axis are at the same point on the real axis and the time elapsed by the phasors in motion is identical. At time t the values of $\Psi_{Tn}(t)$ are $\cos(\omega_n t)$ and of $\Psi_{Tn}(t)^*$ are $\cos[-(\omega_n t)] = \cos(\omega_n t)$. Thus $\Psi_{Tn}(t)\ \Psi_{Tn}(t)^* = [\cos(\omega_n t)]^2 = \{[\cos(2\ \omega_n t)] + 1\}/2$.

In the analysis above the starting point was chosen at point (1, 0 i) as the default choice. In that case both unit phasors started their cycle at the same point on the real axis of the complex plane. But there are infinitely many possible starting points for the two phasors consistent with quantum indeterminacy. Any line parallel to the imaginary axis that intersects the unit circle on the complex plane will provide a single point ((1, 0 i) or (–1, 0 i)) or a pair of separate starting points for the two phasors. In addition the two phasors have two distinct ways to occupy each of those pairs of two points separately. These paired point cases can be mathematized by including C or t_C defined above in Sec. III: $\Psi_{Tn}(t) = \cos(\omega_n t + C) = \cos[\omega_n(t + t_C)]$ and $\Psi_{Tn}(t)^* = \cos[-(\omega_n t + t_C)] = \cos[(\omega_n t + t_C)]$:

$$\Psi_{Tn}(t)\ \Psi_{Tn}(t)^* = \{\cos[\omega_n (t + t_C)]\}^2 = \{\cos[2\ \omega_n (t + t_C)] + 1\}/2. \tag{2}$$

Eq.(2) replaces product $\Psi_{Tn}(t)\ \Psi_{Tn}(t)^* = 1$ found in the literature. The original multiplication of the exponentials is 1 for all t. The solution 1 is not a general solution but is a periodic solution. Invoking Born's rule leads to the general solution $[\cos(\omega_n[(t + t_C)]^2 = \{[\cos 2\ \omega_n (t + t_C) + 1]/2\}$ named here the product density time modulation factor (PDTMF). t_C varies from 0 to T_n randomly for a particular hydrogen atom *introducing a vast quantum indeterminacy for bound states of the hydrogen atom*. The rest of this report will take $t_C = 0$ to keep the analysis minimal as before where quantum indeterminacy applies exactly as above. With $t_C = 0$ Born's product $\Psi_n(r, t)\ \Psi_n(r, t)^*$ from Eq.(1) and Eq.(2) is $[\Psi_{Rn}(r)]^2\ \Psi_{Tn}(t)\ \Psi_{Tn}(t)^*$:

$$\Psi_n(r, t)\ \Psi_n(r, t)^* = [\Psi_{Rn}(r)]^2 \{[\cos(2\ \omega_n\ t) + 1]/2.\} \tag{3}$$

Eq.(3) is the electron statistical probability density at any point in the infinitesimally thin shell of radius r centered near the proton of the hydrogen atom as discussed above. Since $[\Psi_{Rn}(r)]^2$ are real positive functions, Eq.(3) consists entirely of real, positive functions greater than or equal to zero. Eq.(2) varies between 0 and 1. Eq.(3) is a positive real function in coordinate physical space. The energy eigenfunction squared, $[\Psi_{Rn}(r)]^2$, modulates the product $\Psi_n(r, t)\ \Psi_n(r, t)^*$ over the range of r from 0 to the maximum identical value of r at the tip of $\Psi_n(r, t)$ or of $\Psi_n(r, t)^*$. Thus the zeros of statistical probability density of the electron in the hydrogen atom in Eq.(3) fall in the same spatial positions as those of the associated Laguerre functions $\Psi_{Rn}(r)$ in $\Psi_n(r, t)$.

V. The Dynamic Structure of the Hydrogen Atom

$\Psi_n(r, t)\ \Psi_n(r, t)^*$ from Eq.(3) is the statistical probability density at any point in the shell determined by $[\Psi_{Rn}(r)]^2$ at time t. The spherical symmetry of $\Psi_{Rn}(r)$ results in a statistical probability density $PD_n(r, t)$ for the entire spherical shell[11] of area $4\ \pi\ r^2$ as being $4\ \pi\ r^2\ \Psi_n(r, t)\ \Psi_n(r, t)^*$:

$$PD_n(r, t) = 4\ \pi\ r^2\ [\Psi_{Rn}(r)]^2 \{[\cos(2\ \omega_n\ t)] + 1\}]/2. \tag{4}$$

The statistical probability density $PD_n(r, t)$ in Eq.(4) is the statistical electron volume occupation density.[13] $PD_n(r, t)$ waxes and wanes due to the PDTMF (Eq.(2)), $[\cos(2\ \omega_n\ t) + 1]/2$. What is waxing and waning is the statistical probability enclosed in the volume in space defined by the square of energy eigenfunction $[\Psi_{Rn}(r)]^2$. At the instant the probability density of $PD_n(r, t)$ is 0, the electron is very near the proton at the origin. $PD_n(r, t)$ waxes and wanes with twice the explicit frequency ω_n in Eq.(1) starting from the full value of $\Psi_{Rn}(r)^2$ to 0 twice for each cycle of $\cos(2\ \omega_n\ t)$ in time interval T_n. The doubling of frequency to $2\ \omega_n$ is due to the squaring of $\Psi_{Rn}(r)$ during the formation Eq.(3) from Born's product. Every other annular band of $\Psi_{Rn}(r)$ in $\Psi_n(r, t)$ or in $\Psi_n(r, t)^*$ is negative. Thus regions where $\Psi_{Rn}(r) < 0$ become $[\Psi_{Rn}(r)]^2 > 0$ when forming Born's product. $PD_n(r, t)$ retains its unchanging shape as the statistical probability densities wax and wane: the boundaries defined by $[\Psi_{Rn}(r)]^2$ from Eq.(3) remain fixed. This evolving system is similar to the latest experiments that build the statistical diffraction patterns for a beam of electrons beyond a double slit barrier one electron at a time. The quantum world is noisy, but Born's rule finds the structure. Eq.(4) is the statistical dynamic structure of the hydrogen atom.

The maximum probability densities, $PD_n(r, 0)$, can be found as artistic renderings in texts[12,13,14,15] and by searching the internet for hydrogen atom probability density. $PD_n(r, t)$ exhibits a statistical central sphere of probability density governed by $[\Psi_{R1}(r)]^2$ centered at the origin for n = 1. For n = 2 there is a central sphere and a thick shell farther out with density governed by $[\Psi_{R2}(r)]^2$. For n = 3 there is a central sphere and two separate thick shells farther out with density governed by $[\Psi_{R3}(r)]^2$. These results are consistent with the complex statistical domain of $\Psi_n(r, t)$ being in the form of annular patterns. The r-axis crossings have the same value of r for $\Psi_n(r, t)$ as the 0 minima for $PD_n(r, t)$. Using Born's rule forming Eq.(3) extends spherical symmetry to the statistical probability density, $PD_n(r, t)$, in Eq(4). Since the atomic volume defined by $[\Psi_{Rn}(r)]^2$ for s states has spherical symmetry, continuous radiation from the electron is excluded providing for a stable atom.

$PD_n(r, t)$ shows the physical extent of space occupied by an electron as it covers its domain as a statistical physical standing wave. Quantum physics generates this dynamic structure of the hydrogen atom. The statistical perpetual motion oscillatory dynamics of the standing waves in coordinate physical space of bound electron for the duration of the s state creates the world we see. This is an entirely different way of being than anything we are accustomed to observing in the classical world. Electrons with inappropriate eigenfunctions straying from the outside world into the space defined by $PD_n(r, t)$ would be pummeled by a standing wave of frequency of $2\ \omega_n$ by some fraction of the charge of the hydrogen atom's electron. The deeper the penetration by the foreign electron, the bigger the electron charge fraction opposing the invasion provided by $PD_n(r, t)$.

It is clear from Schrödinger's response to Born's rule that his eigenfunctions were not conceived nor expected by him to produce statistical probability distributions. Schrödinger apparently also saw no reason to place $\Psi_n(r, t)$ on the complex plane to see what he had even though this is an essential exercise in quantum physics. The hydrogen atom with its unfamiliar complete complex energy eigenfunctions projected onto the complex plane got lost in the shuffle as well as did the perpetual oscillatory motion of the statistical standing wave patterns of the electron in physical space for any given bound state of the electron in the hydrogen atom.

The signal role of the obscure associated Laguerre functions in the structure of the hydrogen atom is extraordinary. The patterns generated by these functions in the hydrogen atom extend throughout the periodic table[16]. The match of energy eigenvalues of the hydrogen atom with hydrogen spectra clinch the associated Laguerre functions as the energy eigenfunctions with their accompanying spherical symmetry. The finding that these functions applied to hydrogen through Schrödinger's equation is evidence of Schrödinger's deep expertise applying the mathematics of his time. And Schrödinger's equation works well for both bound states and running states of beams of quantum entities as well as other quantum systems. But the prize was hydrogen eigenvalues and atomic stability. Schrödinger opened up a vast exploration of nuclear physics, condensed matter physics, chemistry and more including the cosmic microwave background radiation from free hydrogen atoms in the early cooling universe. On the other hand Born made a significant discovery leading to deeper understanding, Schrödinger's objections not withstanding.

References

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16. For Z = 1 Schrödinger's equation provides accurate predictions for hydrogen's physical properties. In the case of Z = 2 where the helium atom has two electrons, a positive potential energy between the two electrons results in a qualitative and quantitative change of the energy content of the helium atom: The qualitative change is a dynamic stochastically induced spherical symmetry of the positive potential energy. The quantitative change results from the positive potential energy reducing the binding energies of the electrons. The adjusted positive potential energy is included in the Hamiltonian of the time-independent Schrödinger equation. No eigenvalue/eigenfunction solution for the time-independent Schrödinger equation has yet been found for atoms with Z > 1. Nevertheless beyond hydrogen there are 91 consecutive additional atomic species found in Earth's crust. The existence of all atoms with Z > 1 requires spherical symmetry of the presumed energy eigenfunctions for atomic stability as noted in Section V for hydrogen. Such spherically symmetric energy eigenfunctions would also provide atomic spectra for atoms with Z > 1 similar to the spectra of the hydrogen atom; such spectra exist. The spherically symmetric energy eigenfunctions would also provide dynamic standing wave structure of atoms with Z > 1 similar to that of the hydrogen atom; the dynamic standing wave structure of all atoms is consistent with chemistry and condensed matter physics.